Time-consistent and Strategically Supported Cooperation in Dynamic Games

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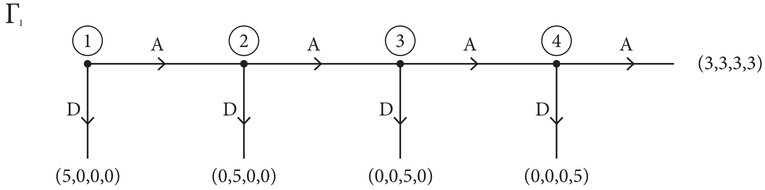
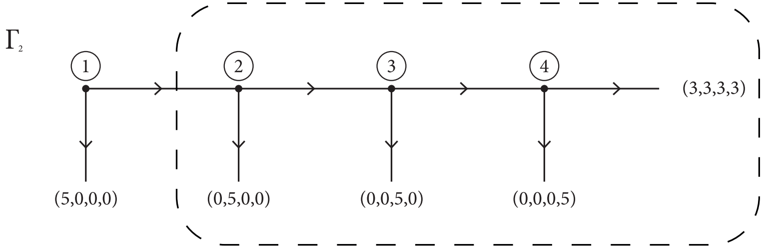
Example.



(D, D, D, D) – NE, (A, A, A, A) – NOT NE

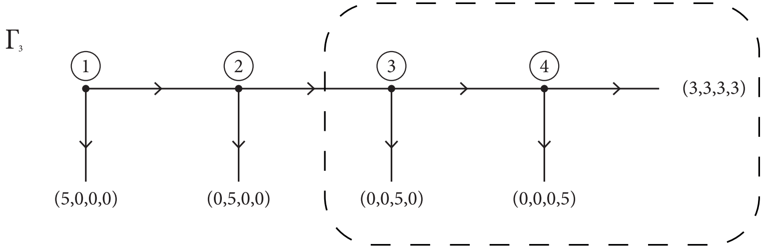
Characteristic Function of the game Γ_1 (C.f. of Γ_1)
$v_1(1,2,3,4) = 12$, $v_1(1,2,3) = 5$, $v_1(1,3,4) = 5$, $v_1(2,3,4) = 0$, $v_1(1,2,4) = 5$,
$v_1(1,2) = 5$, $v_1(1,3) = 5$, $v_1(1,4) = 5$, $v_1(2,3) = 0$, $v_1(2,4) = 0$, $v_1(3,4) = 0$,
$v_1(1) = 5$, $v_1(2) = 0$, $v_1(3) = 0$, $v_1(4) = 0$.

$Sh^1 = \left(\frac{27}{4}, \frac{7}{4}, \frac{7}{4}, \frac{7}{4}\right)$



C.f. of Γ_2
$v_2(1,2,3,4) = 12,\ v_2(1,2,3) = 5,\ v_2(1,3,4) = 5,\ v_2(2,3,4) = 9,$
$v_2(1,2) = 5,\ v_2(1,3) = 0,\ v_2(1,4) = 0,\ v_2(2,3) = 5,\ v_2(2,4) = 5,\ v_2(3,4) = 0,$
$v_2(1) = 0,\ v_2(2) = 5,\ v_2(3) = 0,\ v_2(4) = 0.$

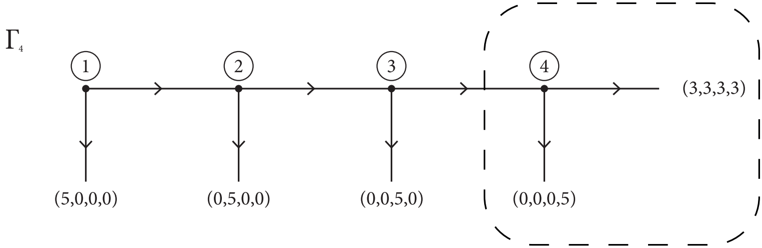
$Sh^2 = \left(\frac{19}{12}, \frac{65}{12}, \frac{30}{12}, \frac{30}{12}\right)$



C.f. of Γ_3

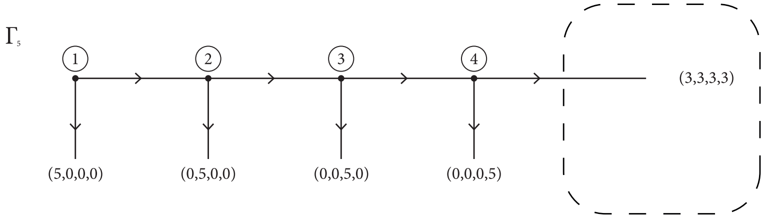
$v_3(1,2,3,4)=12,\ v_3(1,2,3)=5,\ v_3(1,3,4)=9,\ v_3(2,3,4)=9,\ v_3(1,2,4)=0$
$v_3(1,2)=0,\ v_3(1,3)=5,\ v_3(1,4)=0,\ v_3(2,3)=5,\ v_3(2,4)=0,\ v_3(3,4)=6,$
$v_3(1)=0,\ v_3(2)=0,\ v_3(3)=5,\ v_3(4)=0.$

$$Sh^3=\left(1,1,\tfrac{90}{12},\tfrac{30}{12}\right)$$



C.f. of Γ_4
$v_4(1,2,3,4) = 12,\ v_4(1,2,3) = 0,\ v_4(1,3,4) = 9,\ v_4(2,3,4) = 9,\ v_4(1,2,4) = 9$
$v_4(1,2) = 0,\ v_4(1,3) = 0,\ v_4(1,4) = 5,\ v_4(2,3) = 0,\ v_4(2,4) = 5,\ v_4(3,4) = 5,$
$v_4(1) = 0,\ v_4(2) = 0,\ v_4(3) = 0,\ v_4(4) = 5.$

$Sh^4 = \left(\frac{17}{12}, \frac{17}{12}, \frac{17}{12}, \frac{93}{12}\right)$



C.f. of Γ_5

$v_5(1,2,3,4) = 12,\ v_5(1,2,3) = v_5(1,3,4) = v_5(2,3,4) = v_5(1,2,4) = 9$
$v_5(1,2) = v_5(1,3) = v_5(1,4) = v_5(2,3) = v_5(2,4) = v_5(3,4) = 6,$
$v_5(1) = v_5(2) = v_5(3) = v_5(4) = 3.$

$Sh^5 = (3,3,3,3)$

IDP (Imputation Distribution Procedure)

$\beta_k, k = 1, \ldots, 5$

$Sh^1 = \beta_1 + Sh^2, Sh^2 = \beta_2 + Sh^3, \ldots, Sh^4 = \beta_4 + Sh^5$

$\beta_1 = (Sh^1 - Sh^2), \beta_2 = (Sh^2 - Sh^3), \beta_3 = (Sh^3 - Sh^4), \beta_4 = (Sh^4 - Sh^5), \beta_5 = Sh^5$

$\sum_{k=1}^{5} \beta_k = Sh^1, \sum_{k=2}^{5} \beta_k = Sh^2, \sum_{k=3}^{5} \beta_k = Sh^3,$
$\sum_{k=4}^{5} \beta_k = Sh^4, \sum_{k=5}^{5} \beta_k = Sh^5$

$\beta_1 = (\frac{62}{12}, -\frac{44}{12}, -\frac{9}{12}, -\frac{9}{12})$

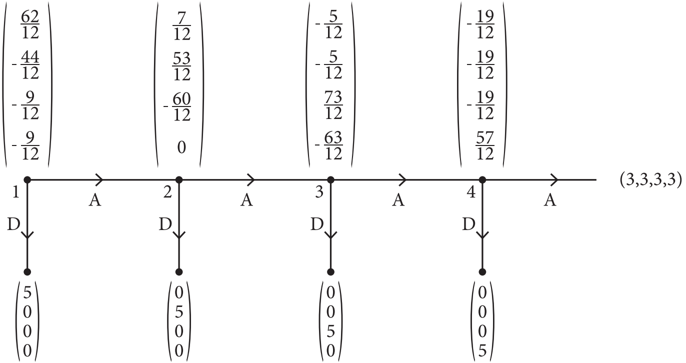
$\beta_2 = (\frac{7}{12}, \frac{53}{12}, -\frac{60}{12}, 0)$

$\beta_3 = (-\frac{5}{12}, -\frac{5}{12}, \frac{73}{12}, -\frac{63}{12})$

$\beta_4 = (-\frac{19}{12}, -\frac{19}{12}, -\frac{19}{12}, \frac{57}{12})$

$\beta_5 = (3, 3, 3, 3)$

Associated Game $\overline{\Gamma}$, and NE Strategically Supported Cooperation



(A, A, A, A) – NE

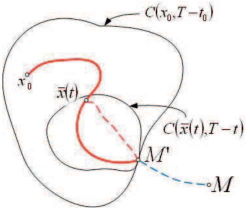
$$
NE \begin{cases} \frac{62}{12} + \frac{7}{12} - \frac{5}{12} - \frac{19}{12} + 3 > 5 \\ \frac{53}{12} - \frac{5}{12} - \frac{19}{12} + 3 > 5 \\ \frac{73}{12} - \frac{19}{12} + 3 > 5 \\ \frac{57}{12} + 3 > 5 \end{cases}
$$

1. Classical control problem.

$$\dot{x} = f(x, u), \quad x \in R^n, u \in U \subset CompR^l,$$
$$x(t_0) = x_0, \quad t \in [t_0, T],$$
$$H(x(T)) = -\rho(x(T), M).$$

$C(x_0, T - t_0)$ – reachability set.
$\bar{x}(t)$ – optimal trajectory.

$\Gamma(x_0, T - t_0), \quad \Gamma(\bar{x}(t), T - t), \quad C(\bar{x}(t), T - t)$



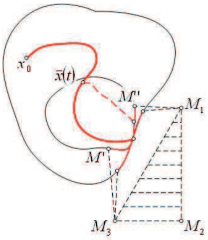
R. Bellmann
Time-consistency, Strong Time-consistency.

2. Multycriterial control.

$$\dot{x} = f(x, u), \quad x \in R^n, u \in U \subset CompR^l,$$
$$x(t_0) = x_0, \quad t \in [t_0, T],$$
$$H(x(T)) = \{H_1(x(T)), \ldots, H_k(x(T))\}.$$

Let $k = 3$, $H_i(x(T)) = -\rho(x(T), M_i)$.

Pareto-optimal solution.
$\bar{x}(t)$ – Pareto-optimal trajectory.
$\Gamma(x_0, T - t_0)$, $\Gamma(\bar{x}(t), T - t)$, $C(x_0, T - t_0)$, $C(\bar{x}(t), T - t)$, $P(x_0, T - t_0)$, $P(\bar{x}(t), T - t)$



TC but not STC

3. Nash bargaining solution in Differential Games.

$$\dot{x} = f(x, u_1, \dots, u_n), \quad x \in R^m, u_i \in U_i \subset CompR^l,$$
$$x(t_0) = x_0, \quad t \in [t_0, T],$$

The payoff of player $i \to H_i(x(T))$,
$\Gamma(x_0, T - t_0)$
$W(x_0, T - t_0; \{i\})$ – the garanteed payoff of player i

NB.

$$\max_{x' \in C(x_0, T - t_0)} \prod_{i=1}^{n} (H_i(x') - W(x_0, T - t_0; \{i\})) = \prod_{i=1}^{n} (H_i(\bar{\bar{x}}) - W(x_0, T - t_0; \{i\}))$$

$\bar{x}(t), \quad x_0 \to \bar{\bar{x}}, \quad \Gamma(\bar{x}(t), T - t), t \in [t_0, T - t_0], \quad W(\bar{x}(t), T - t; \{i\})$

$$\max_{x' \in C(\bar{x}(t), T - t)} \prod_{i=1}^{n} (H_i(x') - W(\bar{x}(t), T - t; \{i\})) = \prod_{i=1}^{n} (H_i(\bar{\bar{x}}(\bar{x}(t))) - W(\bar{x}(t), T - t; \{i\}))$$

$\bar{\bar{x}}(\bar{x}(t)) \neq const \neq \bar{x}$
NB, not TC, not STC

4. Differential Cooperative Game.

Differential Cooperative Game $\Gamma(x_0, T - t_0)$ with prescribed duration $T - t_0$ from the initial position x_0.

$$\dot{x} = f(x, u_1, \ldots, u_n), \quad x \in R^n, u_i \in U_i \tag{1}$$

integral payoff

$$K_i(x_0, T - t_0; u_1, \ldots, u_n) = \int_{t_0}^{T} h_i(x(t))dt, \quad h_i > 0, i = 1, \ldots, n.$$

Cooperative form of $\Gamma(x_0, T - t_0)$.

Cooperative behavior $u^*(t) = \{u_1^*(t), \ldots, u_n^*(t)\}$

$$\begin{aligned}\sum_{i=1}^{n} K_i(x_0, T - t_0; u_1^*, \ldots, u_n^*) &= \\ &= \max_{u_1, \ldots, u_n} \sum_{i=1}^{n} K_i(x_0, T - t_0; u_1, \ldots, u_n) = \\ &= \sum_{i=1}^{n} \int_{t_0}^{T} h_i(x^*(t))dt = v(N; x_0, T - t_0),\end{aligned}$$

$x^*(t)$ – cooperative trajectory.

Characteristic Function in $\Gamma(x_0, T - t_0)$.

$v(S; x_0, T - t_0), \quad S \subset N$,

superadditivity: $v(S_1 \cup S_2; x_0, T - t_0) \geq v(S_1; x_0, T - t_0) + v(S_2; x_0, T - t_0)$,
$S_1 \cap S_2 = \emptyset$.

There are different ways on how to define c. f.

a. Classical: $v(S; x_0, T - t_0) = Val\Gamma_{S,N\setminus S}(x_0, T - t_0)$, where $\Gamma_{S,N\setminus S}(x_0, T - t_0)$ is a zero-sum game played upon the structure of game $\Gamma(x_0, T - t_0)$ between S as player 1 and $N \setminus S$ as player 2.

b. $v(S; x_0, T - t_0) = \sum_{i\in S} K_i(x_0, T - t_0; \bar{u}_S, \bar{u}_{N\setminus S})$, where $(\bar{u}_S, \bar{u}_{N\setminus S})$ is some giver NE in $\Gamma'_{S,N\setminus S}$ played as non zero-sum game over the structure of $\Gamma(x_0, T - t_0)$ between two players: coalition S as player 1 and $N \setminus S$ as player 2

c. $v(S; x_0, T - t_0) = \max_{u_S=\{u_i, i\in S\}} \sum_{i\in S} K_i(x_0, T - t_0; \bar{u}||u_S)$, where $\bar{u} = (\bar{u}_1, \ldots, \bar{u}_n)$ is some fixed NE in $\Gamma(x_0, T - t_0)$.

L. Petrosjan, G. Zaccour Time-consistent Shapley value allocation of pollution cost reduction // Journal of Economics Dunamics & Control, 27 (2003), pp. 381-398.

Let $E(x_0, T - t_0)$ be the imputation set in $\Gamma(x_0, T - t_0)$:

$$E(x_0, T-t_0) = \{\xi = (\xi_i) : \sum_{i=1}^{n} \xi_i = v(N; x_0, T-t_0), \xi_i \geq v(\{i\}; x_0, T-t_0), i \in N\}.$$

Denote by $C^{t-t_0}(x_0), t \in [t_0, T]$ reachable set of the (2).
For each $y \in C^{t-t_0}(x_0)$ consider a subgame $\Gamma(y, T-t)$ of the game $\Gamma(x_0, T-t_0)$, with corresponding c. f. $v(S; y, T-t)$ and set of imputations $E(y, T-t)$.

Definition. A point-to-set mapping $C(y, T-t) \subset E(y, T-t)$ defined for all $y \in C^{t-t_0}(x_0), t \in [t_0, T]$ is call *solution concept* (SC) in the family of subgames $\Gamma(y, T-t)$.

In special cases $C(y, T-t)$ may be a core, NM-solution, Shapley value, nucleous etc.

What happens when the game develops along the cooperative trajectory $x^*(t)$? We pass through current subgame $\Gamma(x^*(t), T-t)$, wilingly or not updating the current SC $\leftrightarrow C(x^*(t), T-t)$.

Imputation Distribution Procedure (IDP).

Let $\bar{\xi} \in C(x_0, T - t_0)$ and $\beta_i(t), i \in N, t \in [t_0, T]$ satisfies the condition

$$\bar{\xi} = \int_{t_0}^{T} \beta_i(t)dt, \quad i \in N, \quad \beta_i \geq 0.$$

$\beta_i(t)$ is called IDP.

Define

$$\bar{\xi}(\theta) = \int_{t_0}^{\theta} \beta_i(t)dt, \quad i \in N, \quad \beta_i \geq 0.$$

Definition. The SC $C(x^*(t), T - t), t \in [t_0, T]$ is called *time-consistent* (TC) if there exist such IDP $\beta(t) = \{\beta_i(t)\}$ that

$$\bar{\xi} - \bar{\xi}(\theta) \in C(x^*(\theta), T - \theta)$$

for all $\theta \in [t_0, T]$.

Definition. The SC $C(x^*(t), T - t), t \in [t_0, T]$ is called *strongly time-consistent* (STC) if there exist such IDP $\beta(t) = \{\beta_i(t)\}$ that

$$\bar{\xi}(\theta) \oplus \bar{C}(x^*(\theta), T - \theta) \subset C(x_0, T - t_0)$$

for all $\theta \in [t_0, T]$. Here $\bar{\xi} \oplus A$ means the set of all possible vectors $\bar{\xi} + \eta$ for all $\eta \in A$.

Consider $C(x^*(t), T-t)$ along $x^*(t), t \in [t_0, T]$. Suppose we can construct a differentiable selector $\xi^t \in C(x^*(t), T-t)$, then we can easily get for $\beta(t)$ the following formula

$$\bar{\xi} = \bar{\xi}(\theta) + \xi^t \quad \rightarrow \quad \bar{\xi} = \int_{t_0}^{\theta} \beta_i(t)dt + \xi^t$$

$$\beta_i(t) = -\frac{d}{dt}\xi^t$$

If ξ^t can be chosen as monotonic nonincreasing (which is very possible since $h_i > 0$, then $\beta_i \geq 0$, and SC is TC.
If the case (for instance) $C(y, T-t)$ is a Shapley value, we get

$$\beta_i(t) = -\sum_{S \subset N, S \ni i} \frac{(n-s)!(s-1)!}{n!} \left[\frac{d}{dt} v(x^*(t), T-t; S) - \right.$$

$$\left. -\frac{d}{dt} v(x^*(t), T-t; S \setminus \{i\}) \right]$$

and we need only differentiability of the value function (c. f.) $v(x, T-t; S)$.

Strategically Supported Cooperation

Continuous time case.

Consider n-person differential game $\Gamma(x_0, T - t_0)$ with prescribed duration and independent motions on the time interval $[t_0, T]$. Motion equations have the form:

$$\dot{x}_i = f_i(x_i, u_i), \quad u_i \in U_i \subset R^{\ell}, x_i \in R^n,$$
$$x_i(t_0) = x_i^0, \quad i = 1, \dots, n. \tag{2}$$

It is assumed that the system of differential equations (2) satisfies all conditions necessary for the existence, prolongability and uniqueness of the solution for any n-tuple of measurable controls $u_1(t), \dots, u_n(t)$.

The payoff of player i is defined as:

$$H_i(x_0, T - t_0; u_1(\cdot), \dots, u_n(\cdot)) = \int_{t_0}^{T} h_i(x(\tau))d\tau,$$

where $h_i(x)$ is a continuous function and $x(\tau) = \{x_1(\tau), \dots, x_n(\tau)\}$ is the solution of (2) when open-loop controls $u_1(t), \dots, u_n(t)$ are used and $x(t_0) = \{x_1(t_0), \dots, x_n(t_0)\} = \{x_1^0, \dots, x_n^0\}$.

Suppose that there exist an n-tuple of open-loop controls $\bar{u}(t) = (\bar{u}_1(t), \ldots, \bar{u}_n(t))$ and the trajectory $\bar{x}(t)$, $t \in [t_0, T]$, such that

$$\max_{u_1(t),\ldots,u_n(t)} \sum_{i=1}^{n} H_i(x_0, T - t_0; u_1(t), \ldots, u_n(t)) =$$
$$= \sum_{i=1}^{n} H_i(x_0, T - t_0; \bar{u}_1(t), \ldots, \bar{u}_n(t)) = \sum_{i=1}^{n} \int_{t_0}^{T} h_i(\bar{x}_i(\tau))d\tau \quad (3)$$

The trajectory $\bar{x}(t) = (\bar{x}_1(t), \ldots, \bar{x}_n(t))$ satisfying (3) we shall call "optimal cooperative trajectory".
Let $N = \{1, \ldots, n\}$ be the set of players. Define in $\Gamma(x_0, T - t_0)$ characteristic function in a classical way:

$$V(x_0, T - t_0; N) = \sum_{i=1}^{n} \int_{t_0}^{T} h_i(\bar{x}_i(\tau))d\tau,$$
$$V(x_0, T - t_0; \emptyset) = 0,$$
$$V(x_0, T - t_0; S) = Val\, \Gamma_{S,N\setminus S}(x_0, T - t_0), \quad (4)$$

where $Val\, \Gamma_{S,N\setminus S}(x_0, T - t_0)$ is a value of zero-sum game played between coalition S acting as first player and coalition $N \setminus S$ acting as player 2, with payoff of player S equal to:

$$\sum_{i\in S} H_i(x_0, T - t_0; u_1(\cdot), \ldots, u_n(\cdot)).$$

Define $L(x_0, T - t_0)$ as imputation set in the game $\Gamma(x_0, T - t_0)$ (see Neumann and Morgenstern (1947)):

$$L(x_0, T - t_0) = \{\alpha = (\alpha_1, \dots, \alpha_n) : \\ \left. \alpha_i \geqslant V(x_0, T - t_0; \{i\}), \quad \sum_{i \in N} \alpha_i = V(x_0, T - t_0; N) \right\}. \quad (5)$$

Regularized game $\Gamma_\alpha(x_0, T - t_0)$. For every $\alpha \in L(x_0, T - t_0)$ define the noncooperative game $\Gamma_\alpha(x_0, T - t_0)$, which differs from the game $\Gamma(x_0, T - t_0)$ only by payoffs defined along optimal cooperative trajectory $\bar{x}(\tau)$, $\tau \in [t_0, T]$.
Let $\alpha \in L(x_0, T - t_0)$. Define the imputation distribution procedure (IDP) (see Petrosjan (1993)) as function $\beta(\tau) = (\beta_1(\tau), \dots, \beta_n(\tau))$, $\tau \in [t_0, T]$ such that

$$\alpha_i = \int_{t_0}^{T} \beta_i(\tau) d\tau. \quad (6)$$

Denote by $H_i^\alpha(x_0, T - t_0; u_1(\cdot), \dots, u_n(\cdot))$ the payoff function in the game $\Gamma_\alpha(x_0, T - t_0)$ and by $x(\tau)$ the corresponding trajectory, then

$$H_i^\alpha(x_0, T - t_0; u_1(\cdot), \dots, u_n(\cdot)) = H_i(x_0, T - t_0; u_1(\cdot), \dots, u_n(\cdot))$$

if there does not exist such $t \in [t_0, T]$ that $x(\tau) = \bar{x}(\tau)$ for $\tau \in (t_0, t]$.

Let $t = \sup\{t' : x(\tau) = \bar{x}(\tau), \tau \in [t_0, t']\}$ and $t > t_0$, then

$$H_i^{\alpha}(x_0, T - t_0; u_1(\cdot), \dots, u_n(\cdot)) =$$
$$= \int_{t_0}^{t} \beta_i(\tau) d\tau + H_i(\bar{x}(t), T - t; u_1(\cdot), \dots, u_n(\cdot)) =$$
$$= \int_{t_0}^{t} \beta_i(\tau) d\tau + \int_{t}^{T} h_i(x(\tau)) d\tau.$$

In a special case, when $x(\tau) = \bar{x}(\tau)$, $\tau \in [t_0, T]$ (if $x(\tau)$ is an optimal cooperative trajectory in the sense of Eq. (3)), we have

$$H_i^{\alpha}(x_0, T - t_0; \bar{u}_1(\cdot), \dots, \bar{u}_n(\cdot)) = \int_{t_0}^{T} \beta_i(\tau) d\tau = \alpha_i.$$

By the definition of payoff function in the game $\Gamma_{\alpha}(x_0, T - t_0)$we get that the payoffs along the optimal trajectory are equal to the components of the imputation $\alpha = (\alpha_1, \dots, \alpha_n)$.
Consider the current subgames (see Neumann and Morgenstern (1947)) — $\Gamma(\bar{x}(t), T - t)$ along $\bar{x}(t)$ and current imputation sets $L(\bar{x}(t), T - t)$. Let $\alpha(t) \in L(\bar{x}(t), T - t)$. Suppose that $\alpha(t)$ can be selected as differentiable function of t, $t \in [t_0, T]$.

Definition 1. The game $\Gamma_\alpha(x_0, T - t_0)$ is called regularization of the game $\Gamma(x_0, T - t_0)$ (α-regularization) if the IDP β is defined in such a way that

$$\alpha_i(t) = \int_t^T \beta_i(\tau) d\tau$$

or

$$\beta_i(t) = -\alpha_i'(t). \tag{7}$$

Theorem 1. In the regularization of the game $\Gamma_\alpha(x_0, T - t_0)$ for every $\varepsilon > 0$ there exist an ε-Nash equilibrium (Nash (1951)) with payoffs $\alpha = (\alpha_1, \ldots, \alpha_i, \ldots, \alpha_n)$.

Proof. The proof is based on actual constraction of the ε-Nash equilibrium in piecewise open-loop (POL) strategies with memory.
Remind the definition of POL strategies with memory in differential game. Denote by $\hat{x}(t)$ any admissible trajectory of the system (2) on the time interval $[t_0, t]$, $t \in [t_0, T]$.
The strategy $u_i(\cdot)$ of player i is called POL if it consists from the pair (a, σ), where σ is a partition of time interval $[t_0, T]$, $t_0 < t_1 < \ldots < t_l = T$ ($t_{k+1} - t_k = \delta > 0$), and a mapping a which corresponds to each point $(\hat{x}(t_k), t_k)$, $t_k \in \sigma$ an open-loop control $u_i(t)$, $t \in [t_k, t_{k+1})$.

Consider a family of associated with $\Gamma(x, T-t)$, but not with $\Gamma_\alpha(x, T-t)$ zero-sum games $\Gamma_{\{i\},N\setminus\{i\}}(x, T-t)$ from the initial position x and duration $T-t$ between the coalition S consisting from a single player i and the coalition $N\setminus\{i\}$ with player's i payoff equal to

$$H_i(x, T-t; u_1(\cdot)\ldots, u_n(\cdot)).$$

The payoff of player $N\setminus\{i\}$ in $\Gamma_{\{i\},N\setminus\{i\}}(x, T-t)$ equals to $(-H_i)$. Let $\hat{u}(x, t; \cdot)$ be the ε-optimal POL strategy of player $N\setminus\{i\}$ in $\Gamma_{\{i\},N\setminus\{i\}}(x, T-t)$. Note, that $\hat{u}(x, t; \cdot) = \{\hat{u}_j(x, t; \cdot)\}$, $j \in N\setminus\{i\}$.
Let $\hat{x}(t) = \{\hat{x}_1(t), \ldots, \hat{x}_n(t)\}$ be the segment of an admissible trajectory of (2) defined on the time interval $[t_0, t]$, $t \in [t_0, T]$. For each $i \in \{1, \ldots, n\}$ define $\bar{t}(i) = \sup\{t_i : \hat{x}_i(t_i) = \hat{\hat{x}}_i(t_i)\}$ and $\bar{\bar{t}}(j) = \min_i \bar{t}(i) = \bar{t}(j)$. $\bar{\bar{t}}(j)$ lies in one of the intervals $[t_k, t_{k+1})$, $k = 0, 1, \ldots, l-1$. Thus, $\bar{t}(i) - t_0$ is the length of the time interval starting from t_0 on which $x_i(t)$ coincides with $\bar{x}_i(t)$ — the i-th component of the cooperative trajectory $\bar{x}(t)$. And $\bar{\bar{t}}(j) - t_0$ is the length of the time interval starting from t_0 on which $x(t)$ coincides with cooperative trajectory $\bar{x}(t)$.

Define the following strategies of player $i \in N$.

$$u_i^*(\cdot) = \begin{cases} \bar{u}_i(t) & \text{for } (\hat{x}(t_k), t_k) \text{ on the optimal cooperative} \\ & \text{trajectory } \bar{x}(t) \;\; (\hat{x}(\tau) = \bar{x}(\tau), \tau \in [t_0, t_k]); \\ \hat{u}_i(\hat{x}(t_{k+1}), t_{k+1}; \cdot) & i\text{-th component of the } \varepsilon/2\text{-optimal POL} \\ & \text{strategy of player } N \setminus \{j\} \text{ in the game} \\ & \Gamma_{\{j\}, N\setminus\{j\}}(x(t_{k+1}), T - t_{k+1}), \text{ if } t_k \leqslant \bar{\bar{t}}(j) < t_{k+1}; \\ \text{arbitrary} & \text{for all other positions.} \end{cases}$$

Show that $u^*(\cdot) = (u_1^*(\cdot), \ldots, u_n^*(\cdot))$ is ε-Nash equilibrium in $\Gamma_\alpha(x_0, T - t_0)$. The following equality holds

$$H_i(x_0, T - t_0; u^*(\cdot)) = H_i(x_0, T - t_0; u_1^*(\cdot), \ldots, u_n^*(\cdot)) = \int_{t_0}^{T} \beta_i(t)dt = \alpha_i. \quad (8)$$

Consider the n-tuple $(u^*(\cdot)||u_i(\cdot))$ where player i changes his strategy $u_i^*(\cdot)$ on $u_i(\cdot)$.

We have to show that

$$H_i(x_0, T - t_0; u^*(\cdot)) \geqslant H_i(x_0, T - t_0; u^*(\cdot)||u_i(\cdot)) - \varepsilon. \quad (9)$$

for all $i \in N$ and all POL $u_i(\cdot)$ of player i.

It is easy to see that when the n-tuple $u^*(\cdot)$ is played the game develops along the optimal trajectory $\bar{x}(t)$. If in $(u^*(\cdot)||u_i(\cdot))$ the trajectory $\bar{x}(t)$ is also realized then (9) will be equality and thus true.

Suppose now that in $(u^*(\cdot)||u_i(\cdot))$ the trajectory $x(t)$ different form $\bar{x}(t)$ is realized. Then let

$$\bar{t} = \inf\{t : \bar{x}(t) \neq x(t)\}.$$

and $\bar{t} \in [t_{k-1}, t_k)$. Since the motion of players are independent we get $\bar{x}_m(t_k) = x_m(t_k)$ for $m \in N \setminus \{i\}$ and $\bar{x}_i(t_k) \neq x_i(t_k)$ (but $\bar{x}_j(t_{k-1}) = x_j(t_{k-1})$ for $j \in N$). Then from the definition of $u^*(\cdot)$ it follows that the players $m \in N \setminus \{i\}$ will use their strategies $\hat{u}_m(\hat{x}(t_k), t_k; \cdot)$ which are $\frac{\varepsilon}{2}$-optimal in a zero-sum game $\Gamma_{\{i\}, N\setminus\{i\}}(x(t_k), T - t_k)$ against the player i which deviates from the optimal trajectory on a time interval $[t_{k-1}, t_k)$.
If the players from the set $N \setminus \{i\}$ will use their strategies $\hat{u}_m(\hat{x}(t_k), t_k; \cdot)$, player i starting from position $x(t_k), t_k$ will get not more than

$$V(x(t_k), T - t_k; \{i\}) + \frac{\varepsilon}{2},$$

where $V(x(t_k), T - t_k; \{i\})$ is the value of the game $\Gamma_{\{i\}, N\setminus\{i\}}(x(t_k), T - t_k)$. Then the total payoff of player i in $\Gamma_\alpha(x_0, T - t_0)$ when the n-tuple of strategies $(u^*(\cdot)||u_i(\cdot))$ is played cannot exceed the amount

$$\int_{t_0}^{t_{k-1}} \beta_i(\tau)d\tau + V(x(t_k), t_k; \{i\}) + \frac{\varepsilon}{2} + \int_{t_{k-1}}^{t_k} h_i(x_i(\tau))d\tau. \qquad (10)$$

But the payoff of player i when the n-tuple $u^*(\cdot)$ is played is equal to

$$\alpha_i = \int_{t_0}^{T} \beta_i(\tau)d\tau = \int_{t_0}^{t_{k-1}} \beta_i(\tau)d\tau + \int_{t_{k-1}}^{T} \beta_i(\tau)d\tau = \int_{t_0}^{t_{k-1}} \beta_i(\tau)d\tau + \alpha_i(t_{k-1}). \tag{11}$$

By the definition of IDP (see (6), (7)), $\alpha_i(t_{k-1}) \in L(\bar{x}(t_{k-1}), T - t_{k-1})$,

$$\int_{t_{k-1}}^{T} \beta_i(\tau)d\tau = \alpha_i(t_{k-1}) \geqslant V(\bar{x}(t_{k-1}), T - t_{k-1}; \{i\}). \tag{12}$$

From the continuity of the function V and continuity of the trajectory $x(t)$ by appropriate choice of $\delta > 0$ ($t_{k+1} - t_k = \delta$) the following inequalities can be guaranteed:

$$|V(\bar{x}(t_{k-1}), T - t_{k-1}; \{i\}) - V(x(t_k), T - t_k; \{i\})| < \frac{\varepsilon}{4},$$

$$\int_{t_{k-1}}^{T} \beta_i(\tau)d\tau = \alpha_i(t_{k-1}) \geqslant V(x(t_k), T - t_k; \{i\}) - \frac{\varepsilon}{4}.$$

Compare $\alpha_i(t_{k-1})$ and $V(x(t_k), t_k; \{i\}) + \frac{\varepsilon}{2} + \int_{t_{k-1}}^{t_k} h_i(x_i(\tau))d\tau$. By choosing $\delta = t_{k+1} - t_k$ sufficiently small one can achieve that the integral $\int_{t_{k-1}}^{t_k} h_i(x_i(\tau))d\tau$ will be also small (less than $\varepsilon/4$).

Adding to both sides of (12) the amount $\int_{t_0}^{t_{k-1}} \beta_i(\tau)d\tau$ and using the previous inequality we get

$$\alpha_i = \int_{t_0}^{t_{k-1}} \beta_i(\tau)d\tau + \alpha_i(t_{k-1}) \geqslant \int_{t_0}^{t_{k-1}} \beta_i(\tau)d\tau + V(\bar{x}(t_{k-1}), T - t_{k-1}; \{i\}) \geqslant$$
$$\geqslant \int_{t_0}^{t_{k-1}} \beta_i(\tau)d\tau + V(x(t_k), T - t_k; \{i\}) - \frac{\varepsilon}{4}$$
$$\geqslant \int_{t_0}^{t_{k-1}} \beta_i(\tau)d\tau + V(x(t_k), T - t_k; \{i\}) - \frac{\varepsilon}{4} + \int_{t_{k-1}}^{t_k} h_i(\tau)d\tau - \frac{\varepsilon}{4}$$
$$\geqslant \int_{t_0}^{t_{k-1}} \beta_i(\tau)d\tau + V(x(t_k), T - t_k; \{i\}) + \int_{t_{k-1}}^{t_k} h_i(\tau)d\tau - \frac{\varepsilon}{2}$$
$$\geqslant \int_{t_0}^{t_{k-1}} \beta_i(\tau)d\tau + V(x(t_k), T - t_k; \{i\}) + \int_{t_{k-1}}^{t_k} h_i(\tau)d\tau +$$
$$+\frac{\varepsilon}{2} - \frac{\varepsilon}{2} - \frac{\varepsilon}{2}. \tag{13}$$

Here first four addends in the right part of the inequality constitute the upper bound of player i payoff when $(u^*(\cdot)||u_i^*(\cdot))$ is played. But α_i is the payoff of player i when $u^*(\cdot)$ is played, and we get

$$\begin{aligned} H_i(x_0, T - t_0; u^*(\cdot)) = \alpha_i \geqslant \\ \geqslant \int_{t_0}^{t_{k-1}} \beta_i(\tau)d\tau + V(x(t_k), T - t_k; \{i\}) + \int_{t_{k-1}}^{t_k} h_i(\tau)d\tau + \frac{\varepsilon}{2} - \varepsilon \geqslant \\ \geqslant H_i(x_0, T - t_0; u^*(\cdot)||u_i(\cdot)) - \varepsilon \end{aligned} \tag{14}$$

and we get (9). The theorem is proved. □

This means that the cooperative solution (any imputation) can be strategically supported in a regularized game $\Gamma_\alpha(x_0, T - t_0)$ (realized in a specially constructed Nash equilibrium) by the Nash equilibrium $u^*(\cdot)$ defined in the Theorem 1.

Discrete time case.

In what follows as basic model we shall consider the game in extensive form with perfect information.

Definition 2. A game tree is a finite oriented treelike graph K with the root x_0. We shall use the following notations. Let x be some vertex (position). We denote by $K(x)$ a subtree K with the root in x. We denote by $Z(x)$ immediate successors of x. The vertices y, directly following after x, are called alternatives in x ($y \in Z(x)$). The player who makes a decision in x (who selects the next alternative position in x), will be denoted by $i(x)$. The choice of player $i(x)$ in position x will be denoted by $\bar{x} \in Z(x)$.

Let $N = \{1, \dots, n\}$ — be the set of all players in the game.

Definition 3. A game in extensive form with perfect information (see Kuhn (1953)) $G(x_0)$ is a graph tree $K(x_0)$, with the following additional properties:

- The set of vertices (positions) is split up into $n+1$ subsets $P_1, P_2, \dots, P_{n+1}$, which form a partition of the set of all vertices of the graph tree K. The vertices (positions) $x \in P_i$ are called players i personal positions, $i = 1, \dots, n$; vertices (positions) $x \in P_{n+1}$ are called terminal positions.
- In each final vertex (position) the system of real numbers $h(w) = (h_1(w), \dots, h_n(w))$, $w \in P_{n+1}$, $h_i(w) \geqslant 0$, $i = 1, \dots, n$ is defined. Where $h_i(w)$ is the payoff of player i in the final vertex (position).

Definition 4. A strategy of player i is a mapping $U_i(\cdot)$, which associate to each position $x \in P_i$ a unique alternative $y \in Z(x)$.
As in the previous case denote by $H_i(x; u_1(\cdot), \dots, u_n(\cdot))$ the payoff function od player $i \in N$ in the subgame $G(x)$ starting from the position x.

$$H_i(x; u_1(\cdot), \dots, u_n(\cdot)) = h_i(x'_l)$$

where $x'_l \in P_{n+1}$ is the last vertex (position) in the path $x = (x'_1, x'_2, \dots, x'_l)$ realized in the subgame $G(x)$, when the n-tuple of strategies $(u_1(\cdot), \dots, u_n(\cdot))$ is played.
Denote by $\bar{u}(\cdot) = (\bar{u}_1(\cdot), \dots, \bar{u}_n(\cdot))$ the n-tuple of strategies and the trajectory (path) $\bar{x} = (\bar{x}_0, \bar{x}_1, \dots, \bar{x}_m)$, $\bar{x}_m \in P_{n+1}$ such that

$$\max_{u_1(\cdot), \dots, u_n(\cdot)} \sum_{i=1}^{n} H_i(x_0; u_1(\cdot), \dots, u_n(\cdot)) =$$

$$= \sum_{i=1}^{n} H_i(x_0; \bar{u}_1(\cdot), \dots, \bar{u}_n(\cdot)) = \sum_{i=1}^{n} h_i(\bar{x}_m). \qquad (15)$$

The path $\bar{x} = (\bar{x}_0, \ldots, \bar{x}_m)$ satisfying Eq. (15) we shall call "optimal cooperative trajectory".
Define in $G(x_0)$ characteristic function in a classical way

$$
\begin{aligned}
&V(x_0; N) = \sum_{i=1}^{n} h_i(\bar{x}_m),\\
&V(x_0; \emptyset) = 0,\\
&V(x_0; S) = Val\ \Gamma_{S, N \setminus S}(x_0),
\end{aligned}
$$

where $Val\ \Gamma_{S, N\setminus S}(x_0)$ is a value of zero-sum game played between coalition S acting as first player and coalition $N \setminus S$ acting as player 2, with payoff of player S equal to

$$
\sum_{i \in S} H_i(x_0; u_1(\cdot), \ldots, u_n(\cdot)).
$$

Define $L(x_0)$ as imputation set in the game $G(x_0)$.

$$
L(x_0) = \left\{ \alpha = (\alpha_1, \ldots, \alpha_n) : \alpha_i \geqslant V(x_0; \{i\}), \quad \sum_{i \in N} \alpha_i = V(x_0; N) \right\}.
$$

Regularized game $G_\alpha(x_0)$. For every $\alpha \in L(x_0)$ define the noncooperative game $G_\alpha(x_0)$, which differs from the game $G(x_0)$ only by payoffs defined along optimal cooperative path $\bar{x} = (\bar{x}_0, \ldots, \bar{x}_m)$. Let $\alpha \in L(x_0)$. Define the imputation distribution procedure (IDP) as function $\beta_k = (\beta_1(k), \ldots, \beta_n(k))$, $k = 0, 1, \ldots, m$ such that

$$\alpha_i = \sum_{k=0}^{m} \beta_i(k). \tag{16}$$

Define by $H_i^\alpha(x_0; u_1(\cdot), \ldots, u_n(\cdot))$ the payoff function in the game $G_\alpha(x_0)$ and by $\bar{x} = \{\bar{x}_0, \ldots, \bar{x}_m\}$ the cooperative path

$$H_i^\alpha(x_0; u_1(\cdot), \ldots, u_n(\cdot)) = H_i(x_0; u_1(\cdot), \ldots, u_n(\cdot))$$

for all $u_1(\cdot), \ldots, u_n(\cdot)$ such that the path $x = \{x_0, \ldots, x_m\}$ differs from $\bar{x} = \{\bar{x}_0, \ldots, \bar{x}_m\}$, and

$$H_i^\alpha(x_0; \bar{u}_1(\cdot), \ldots, \bar{u}_n(\cdot)) = \alpha_i.$$

By the definition of the payoff function in the game $G_\alpha(x_0)$ we get that the payoffs along the optimal cooperative trajectory are equal to the components of the imputation $\alpha = (\alpha_1, \ldots, \alpha_n)$.

Consider current subgames $G(\bar{x}_k)$ along the optimal path $\bar{x}$ and current imputation sets $L(\bar{x}_k)$. Let $\alpha^k \in L(\bar{x}_k)$.
Definition 5. The game $G_\alpha(x_0)$ is called regularization of the game $G(x_0)$ (α-regularization) if the IDP β is defined in such a way that

$$\alpha_i^k = \sum_{j=k}^{m} \beta_i(j)$$

or $\beta_i(k) = \alpha_i^k - \alpha_i^{k+1}$, $i \in N$, $k = 0, 1, \ldots, m-1$, $\beta_i(m) = \alpha_i^m$, $\alpha_i^0 = \alpha_i$.
Theorem 2. In the regularization of the game $G_\alpha(x_0)$ there exist a Nash equilibrium with payoffs $\alpha = (\alpha_1, \ldots, \alpha_n)$.
Proof. Along the cooperative path we have

$$\alpha_i^k \geqslant V(\bar{x}_k; \{i\}), \quad i \in N, k = 0, 1, \ldots, m.$$

since $\alpha^k = (\alpha_1^k, \ldots, \alpha_n^k) \in L(\bar{x}_k)$ is an imputation in $G(\bar{x}_k)$ (note that here $V(\bar{x}_k; \{i\})$ is computed in the subgame $G(\bar{x}_k)$ but not $G_\alpha(\bar{x}_k)$). In the same time

$$\alpha_i^k = \sum_{j=k}^{m} \beta_i(j)$$

and we get

$$\sum_{j=k}^{m} \beta_i(j) \geqslant V(\bar{x}_k; \{i\}), \quad i \in N, k = 0, 1, \dots, m. \tag{17}$$

But $\sum_{j=k}^{m} \beta_i(j)$ is the payoff of player i in the subgame $G_\alpha(\bar{x}_k)$ along the cooperative path, and from (17) using the arguments similar to those in the proof of Theorem 1 one can construct the Nash equilibrium with payoffs $\alpha = (\alpha_1, \dots, \alpha_n)$ and resulting cooperative path $\bar{x} = (\bar{x}_0, \dots, \bar{x}_m)$.

Example. In this example as an imputation we shall consider Shapley value [Shapley (1953)]. Using the proposed regularization of the game we shall see that there exist a Nash equilibrium with payoffs equal to the components of the Shapley value.

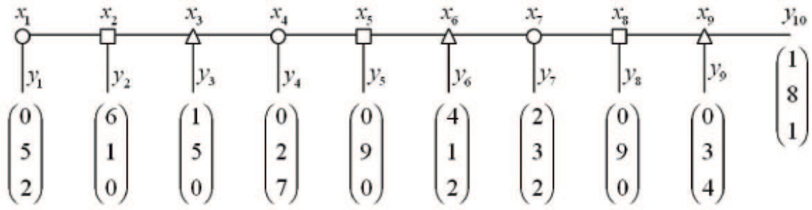


Fig. 1. Game $G(x_0)$

In the game $G(x_0)$, $N = \{1, 2, 3\}$, $P_1 = \{x_1, x_4, x_7\}$, $P_2 = \{x_2, x_5, x_8\}$, $P_3 = \{x_3, x_6, x_9\}$, $P_4 = \{y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9, y_{10}\}$. $h(y_1) = (0, 5, 2)$, $h(y_2) = (6, 1, 0)$, $h(y_3) = (1, 5, 0)$, $h(y_4) = (0, 2, 7)$, $h(y_5) = (0, 9, 0)$, $h(y_6) = (4, 1, 2)$, $h(y_7) = (2, 3, 2)$, $h(y_8) = (0, 9, 0)$, $h(y_9) = (0, 3, 4)$, $h(y_{10}) = (1, 8, 1)$. The cooperative path is $\bar{x} = \{\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4, \bar{x}_5, \bar{x}_6, \bar{x}_7, \bar{x}_8, \bar{x}_9, \bar{y}_{10}\}$.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	y_{10}
$V(x;\{1\})$	0	0	0	0	0	2	2	0	0	1
$V(x;\{2\})$	2	2	2	2	9	1	3	9	3	8
$V(x;\{3\})$	0	0	0	0	0	2	0	0	4	1
$V(x;\{1,2\})$	7	7	6	9	9	5	9	9	3	9
$V(x;\{2,3\})$	7	9	9	9	9	5	5	9	9	9
$V(x;\{1,3\})$	6	6	6	7	0	6	4	0	4	2
$V(x;\{1,2,3\})$	10	10	10	10	10	10	10	10	10	10
$Sh(x;\{1\})$	$\frac{17}{6}$	$\frac{13}{6}$	$\frac{12}{6}$	$\frac{16}{6}$	$\frac{2}{6}$	$\frac{22}{6}$	$\frac{24}{6}$	$\frac{2}{6}$	$\frac{2}{6}$	1
$Sh(x;\{2\})$	$\frac{26}{6}$	$\frac{28}{6}$	$\frac{27}{6}$	$\frac{28}{6}$	$\frac{56}{6}$	$\frac{16}{6}$	$\frac{30}{6}$	$\frac{56}{6}$	$\frac{26}{6}$	8
$Sh(x;\{3\})$	$\frac{17}{6}$	$\frac{19}{6}$	$\frac{21}{6}$	$\frac{16}{6}$	$\frac{2}{6}$	$\frac{22}{6}$	$\frac{6}{6}$	$\frac{2}{6}$	$\frac{32}{6}$	1

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	y_{10}
$\beta_1(j)$	$\frac{2}{6}$	$\frac{4}{6}$	$\frac{1}{6}$	$\frac{14}{6}$	$-\frac{28}{6}$	$-\frac{2}{6}$	$\frac{22}{6}$	0	$-\frac{4}{6}$	1
$\beta_2(j)$	$-\frac{1}{6}$	$-\frac{2}{6}$	$\frac{1}{6}$	$-\frac{28}{6}$	$\frac{40}{6}$	$-\frac{14}{6}$	$-\frac{28}{6}$	$\frac{30}{6}$	$-\frac{22}{6}$	8
$\beta_3(j)$	$-\frac{1}{6}$	$-\frac{2}{6}$	$-\frac{2}{6}$	$\frac{14}{6}$	$-\frac{20}{6}$	$\frac{16}{6}$	$\frac{4}{6}$	$-\frac{30}{6}$	$\frac{26}{6}$	1

It can be easily seen that the inequality (17)

$$\sum_{j=k}^{m} \beta_i(j) \geqslant V(\bar{x}_k; \{i\})$$

for $i \in N$ holds in this case.

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